LEBANESE AMERICAN UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND MATHEMATICS
MTH 201 - CALCULUS III

Exam-I, Fall 2014

Duration: 75 minutes

<u>*INSTRUCTIONS*</u>: *This exam consists of 7 pages and 5 problems. Check that none is missing. Answer the questions in the space provided for each problem; if more space is needed, you may use the back pages. To receive full credits, you have to justify your answers.*

Student's Name: Solutions

Grading scheme
(Keep it empty)

Question 1	/10
Question 2	/30
Question 3	/30
Question 4	/10
Question 5	/20
Total	/100

1. **[10 Points]** Use partial fractions and evaluate the following integral

$$\int \frac{4x^2-5x+3}{x(x-1)^2}\,dx = \int \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\,dx$$

$$= A\ln|x| + B\ln|x-1| - \frac{C}{x-1} + K$$

$\Rightarrow$

$$A(x-1)^2 + Bx(x-1) + Cx = 4x^2 - 5x + 3.$$

$$A(x^2-2x+1) + Bx^2 - Bx + Cx = 4x^2 - 5x + 3$$

$$(A+B)x^2 + (-2A - B + C)x + A = 4x^2 - 5x + 3$$

$$\begin{cases} A+B = 4 \\ -2A - B + C = -5 \\ A = 3 \end{cases}$$

$A=3$
$B=1$
$C=2$

Solve for A, B, C and get:

$$\boxed{A\ln|x| + B\ln|x-1| - \frac{C}{x-1} + K}$$

$= \boxed{\cancel{4\ln|x| - \frac{3}{x-1} + K}}$ $\quad 3\ln|x| + \ln|x-1| - \frac{2}{x-1} + K$

2. **[30 Points]** Calculate the following improper integrals

$$\int_1^e \frac{1}{x(\ln x)^2}\, dx = \qquad u = \ln x.$$

$$\lim \int \frac{du}{u^2} = \lim_{t \to 1} \left[-\frac{1}{\ln x} \Big|_{t^+}^{e} \right] \rightarrow \infty$$

$$= -\frac{1}{\ln e} + \frac{1}{\ln 1}$$

$$= -1 + \infty$$

$\Rightarrow$ diverges.

$-1 + 0 \Rightarrow 6$

$$\int_0^{1} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx = \qquad \int u\, du = \frac{u^2}{2}.$$

$$\lim_{t \to 1^-} \left[\frac{(\sin^{-1} x)^2}{2} \Big|_0^t \right] = \frac{\pi^2/4}{2} = \pi^2/8$$

$$\int_{-\infty}^{\infty} \frac{1}{\cosh x}\,dx = \int \frac{2}{e^{x}+e^{-x}}\,dx \quad \text{even fn.}$$

$$= 2\int_0^{\infty} \frac{2\,dx}{e^{x}+e^{-x}} = 2\int \frac{2\,dx}{e^{x}+\frac{1}{e^{x}}} = 2\int \frac{2e^{x}}{e^{2x}+1}\,du$$

$$\boxed{\lim_{t\to\infty} 4\tan^{-1}(e^{x})\Big|_0^{t}} = 4\cdot\tan^{-1}\infty - 4\tan^{-1}0$$

$$= 4\cdot\frac{\pi}{2}$$

$$= 2\pi$$

3. **[30 Points]** Determine whether the following improper integrals converge or diverge. Justify your answer and precise the test your are using.

$$\int_0^{\infty} \frac{|\cos x|}{(e^{x}+1)^2}\,dx = \int_0^{1} + \int_1^{\infty}$$

$$= \text{proper} + \int_1^{\infty} \frac{|\cos x|}{(e^{x}+1)^2}\,dx$$

$$< \int_1^{\infty} \frac{1}{x^2}\,dx \quad \text{conv.}$$

$\Rightarrow$ int $=$ K + conv. $\longrightarrow$ conv.

$$\int_1^\infty \frac{\sqrt{x}}{\sqrt{x^4+4}}\,dx \approx \int_1^{\infty} \frac{x^{1/2}}{x^2}\,dx = \int_1^{\infty} \frac{dx}{x^{1.5}}$$

p-int. $p > 1$

$\Rightarrow$ conv.

$$\int_1^\infty \frac{\ln x}{x^2}\,dx \qquad \ln x < x^{0.5}$$

$$\frac{\ln x}{x^2} < \frac{x^{0.5}}{x^2} = \frac{1}{x^{1.5}}.$$

$$\therefore \quad < \underbrace{\Sigma \frac{1}{x^{1.5}}\,dx}_{\text{conv.}}$$

$\longrightarrow$ conv. by DCT (smaller than conv.)

4. **[10 Points]**

Find the value of a so that the given integral converges

$$\int_3^{\infty} \left(\frac{a}{x-1} + \frac{1}{x+1} \right) \, dx$$

$$\frac{a}{x-1} + \frac{1}{x+1} = \frac{a(x+1) + 1(x-1)}{(x-1)(x+1)}$$

$$= \frac{(a+1)x + a - 1}{(x-1)(x+1)}$$

If $a = -1 \Rightarrow \approx \frac{1}{x^a} \Rightarrow$ conv.

If $a \neq -1 \Rightarrow \approx \frac{1}{x} \Rightarrow$ div.

$\Rightarrow$ Answer $= \boxed{a = -1}$

5. **[20 Points]** Determine if the following sequences converge or diverge.

(a) $a_n = \dfrac{(\ln n)^7}{\sqrt{n}}$. Hint: Use the sandwich theorem.

$$= \frac{\text{slow}}{\text{fast}} \longrightarrow 0$$

(b) $b_n = \frac{n!}{6^n + 8^n} \approx \frac{n!}{8^n} = \frac{\text{fast}}{\text{slow}} \to \infty$

(c) $c_n = \left(\frac{3}{n}\right)^{\frac{3}{n}} \Rightarrow$ consider

$$y = \left(\frac{3}{n}\right)^{3/n}$$

$$\Rightarrow \ln y = \frac{3}{n} \ln(3/n)$$

$$= \frac{3[\ln 3 - \ln n]}{n} =$$

$$= \frac{\text{slow}}{\text{fast}} \longrightarrow 0.$$

$\ln y \to 0$

$$\Rightarrow \boxed{y \longrightarrow 1}$$